# Improved Mode-Superposition Technique for Modal Frequency Response Analysis of Coupled Acoustic-Structural Systems

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**A new formulation is derived for obtaining the modal frequency response (MFR) of a coupled acoustic-structural system by using an orthogonality condition of a coupled system that was presented earlier by the authors. An improved compensation technique is then proposed for compensating for the effect of truncated modes, which are the lower and/or higher modes beyond the frequency domain of an MFR analysis. A coupled acoustic-box model is used in the analysis to confirm the applicability of the proposed method to actual coupled analysis problems. It is demonstrated that the use of this technique greatly improves convergence, and makes it possible to determine the frequency response with better accuracy than the mode-displacement method, the mode-acceleration method, or Hansteen-Bell's method. Moreover, it does not increase the computational effort.**

## Nomenclature

$e_i$ = error in the new method caused by neglecting $i$th mode
$e_i^a$ = error in the mode-acceleration method caused by neglecting $i$th mode
$e_i^s$ = error in the mode-displacement method caused by neglecting $i$th mode
$F$ = amplitude of the excitation force
$F_i$ = $i$th modal force for MFR
$f$ = excitation force vector
$\bar{f}$ = $\Sigma_{i=1}^{n} M \phi_i f_i$
$f_a$ = excitation force vector that represents interior acoustic sources
$f_h$ = $f - \bar{f} = (I - \Sigma_1^n M \phi_i \bar{\phi}_i^T) f$, residual force
$f_i$ = $i$th modal force
$f_s$ = excitation force vector acting on the structure
$G$ = residual flexibility matrix
$I$ = unit matrix
$K$ = stiffness matrix of the coupled system
$K_{aa}$ = stiffness matrix of the sound field
$K_{sa}$ = coupling term matrix
$K_{ss}$ = stiffness matrix of the structure
$k_i$ = $i$th modal stiffness of the coupled system
$M$ = mass matrix of the coupled system
$M_{aa}$ = mass matrix of the sound field
$M_{as}$ = $-K_{sa}^T$
$M_{ss}$ = mass matrix of the structure
$m$ = number of the lowest-frequency mode used in MFR analysis
$m_i$ = $i$th modal mass of the coupled system
$N$ = total number of DOF of the system
$n$ = number of the highest-frequency mode used in MFR analysis
$Q_i$ = modal displacement coordinate for MFR
$Q_i^d$ = $Q_i(\Omega^2 - \omega_c^2)/(\omega_i^2 - \omega_c^2)$
$q_i$ = modal displacement coordinate
$U$ = modal frequency response (MFR) of the coupled system
$U_d$ = $\Sigma_{i=m}^{n} \phi_i Q_i^d$
$U_r$ = residual MFR in the new method
$U_r'$ = approximation of $U_r$
$U_s$ = $(K - \omega_c^2 M)^{-1} F$
$u$ = nodal displacement vector of the coupled system
$\bar{u}$ = $\Sigma_{i=1}^{n} \phi_i q_i$
$u_a$ = sound pressure vector
$u_h$ = residual response in Hansteen-Bell's method
$u_s$ = nodal displacement vector of the structure
$\beta$ = $\omega_c/\Omega$, a factor
$\lambda_i$ = $i$th eigenvalue of the coupled system
$\phi_i$ = $i$th right eigenvector of the coupled system
$\phi_{ai}$ = component of $\phi_i$ relative to $u_a$
$\phi_{si}$ = component of $\phi_i$ relative to $u_s$
$\bar{\phi}_i$ = $i$th left eigenvector of the coupled system
$\Omega$ = frequency of the excitation force
$\omega_a$ = the lower limit of the frequency domain
$\omega_b$ = the upper limit of the frequency domain
$\omega_c$ = a given frequency $\omega_c \in [\omega_a, \omega_b]$
$\omega_i$ = $i$th natural frequency of the coupled system
$\ddot{}$ = second derivative with respect to time
$|x|$ = norm of vector $x$ or absolute value of constant $x$

## Introduction

SINCE interior noise has a strong effect on vehicle acceptance, it is particularly important to be able to estimate noise levels accurately by means of simulation at the design stage. Such needs have prompted extensive research into simulation techniques using a coupled acoustic-structural finite element model.[1-7] In the finite element model of a coupled acoustic-structural system, the coefficient matrices are not symmetrical. This means that analytical techniques that are dependent on the symmetry of the coefficient matrices, e.g., the mode-superposition technique, cannot be applied to the coupled system in a straightforward manner. To overcome this problem, researchers have tried to symmetrize the coefficient matrices through coordinate transformations.[2-5] However, these coordinate transformations cause the analytical equations to lose their original physical meanings, which are very important in many analyses; e.g., in a system-parameter

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identification problem. Coordinate transformations also increase the computational expense. To overcome these shortcomings, this paper deals with an investigation into the modal frequency response (MFR) method, which is treated using the original asymmetrical coefficient matrices of the coupled system.

As the coefficient matrices of a coupled acoustic-structural system are not symmetrical, the conventional orthogonality conditions obtained in structural dynamics do not hold. For this reason, the orthogonality and normalization conditions of a coupled system were derived in Ref. 5 for calculating the eigenmode sensitivities of a coupled system. Those conditions are used in this paper to derive an analytical formulation for finding the MFR of coupled acoustic-structural systems. It is shown that this formula is identical in form to the formulation used in structural dynamics, although the calculations of the coefficients are different.

One of the advantages of the mode-superposition method is that only a limited number of modes need be included in the analysis. In a large-scale problem, the number of modes used in the analysis is typically far smaller than the total number of variables of a finite element model. However, as pointed out in the structural analyses done in previous research,[8-10] using a standard mode-superposition technique (mode-displacement method) a few modes can yield accurate displacements, but the stress resultants may, for the same number of modes, be quite unacceptable. In coupled acoustic-structural analyses, we have discovered that using a mode-displacement method can yield the structural displacement responses of a coupled system with good accuracy when only a few of the modes of the coupled system are employed, but the sound pressure responses often may be incorrect. This can be explained by the fact that sound pressures are similar in physical meaning to stresses.

As is well known in structural dynamics, the "mode-acceleration method" can be used to correct inaccuracies that occur when higher-frequency modes are ignored.[9] As pointed out by Hansteen and Bell,[10] if the loading frequency is low, the effect of the higher modes can be approximated by a static analysis, i.e., the inaccuracies resulting from the truncating of the higher modes can be corrected by means of static compensation. In this paper, it is proved that Hansteen-Bell's method is equivalent to the mode-acceleration method, which was first proposed by Williams in 1945.[11] Using the orthogonality and normalization conditions presented in Ref. 5, the mode-acceleration method and Hansteen-Bell's method can be extended to coupled systems.

Essentially, in a coupled acoustic-structural problem, relatively higher-frequency structural modes are related to lower-frequency acoustic modes. For example, in an actual vehicle interior noise problem, 50 Hz may correspond to the first acoustic resonance frequency, but it may correspond to the fortieth natural frequency of the body structure.[4,5] Accordingly, to conduct a mode-superposition calculation efficiently, it is necessary to truncate not only the higher modes but also the lower ones, especially in an interior acoustic resonance analysis. However, in the mode-acceleration method (also Hansteen-Bell's method), if lower modes are truncated, the solution turns out to be much less accurate than the result obtained with the mode-displacement method.

In this paper, a new compensation technique is presented whereby the mode-acceleration method and Hansteen-Bell's static compensation are improved by using a new quasistatic compensation. It is shown that the new technique not only applies to truncated higher modes but also to truncated lower modes, provided the modes are beyond the frequency domain of an MFR analysis. It is demonstrated that the use of this technique achieves better calculation accuracy than the mode-displacement method, the mode-acceleration method, or the Hansteen-Bell's method, even when only higher modes are truncated. In addition, it does not increase the computational expense.

The validity of the proposed method is verified by applying it to a coupled acoustic-box structural model.

## Eigenvalue Problem of a Coupled System

This discussion will examine the following finite element equations for treating a coupled acoustic-structural system[2-5]

$$M\ddot{u} + Ku = f \tag{1}$$

where

$$K = \begin{bmatrix} K_{ss} & K_{sa} \\ 0 & K_{aa} \end{bmatrix}, \qquad M = \begin{bmatrix} M_{ss} & 0 \\ M_{as} & M_{aa} \end{bmatrix}$$
$$u = \begin{Bmatrix} u_s \\ u_a \end{Bmatrix}, \qquad f = \begin{Bmatrix} f_s \\ f_a \end{Bmatrix} \tag{2}$$

and $K_{sa} = -M_{as}^T$.

The following eigenvalue problem is obtained from Eq. (1):

$$(K - \omega_i^2 M)\phi_i = 0 \tag{3}$$

Matrices $K$ and $M$ of Eq. (3) are not symmetrical and so the following orthogonality conditions do not apply:

$$\phi_i^T K\phi_j = 0, \qquad \phi_i^T M\phi_j = 0 \qquad \text{for } i \neq j \tag{4}$$

This means that the conventional mode-superposition method cannot be applied to coupled systems. For this reason, the revised orthogonality and normalization conditions were derived in Ref. 5 (see Appendix).

### Orthogonality Condition

The revised orthogonality condition applies to the acoustic-structural system of Eq. (3)

$$\bar{\phi}_i^T K\phi_j = 0, \qquad \bar{\phi}_i^T M\phi_j = 0 \qquad \text{for } i \neq j \tag{5}$$

where $\bar{\phi}_i$ is called the left eigenvector.

When eigenfrequency $\omega_i$ is not zero

$$\bar{\phi}_i = \begin{Bmatrix} \phi_{si} \\ \dfrac{1}{\omega_i^2}\phi_{ai} \end{Bmatrix} \tag{6}$$

When eigenfrequency $\omega_i$ is zero

$$\bar{\phi}_i = \begin{Bmatrix} 0 \\ \phi_{ai} \end{Bmatrix} \qquad \text{for } \phi_{ai} \neq 0$$
$$\bar{\phi}_i = \begin{bmatrix} I \\ K_{aa}^{-1}M_{as} \end{bmatrix}\{\phi_{si}\} \qquad \text{for } \phi_{ai} = 0 \tag{7}$$

### Normalization Condition

The M-normalization condition of the eigenvector $\phi_i$ for a coupled system is given as follows:

When the eigenfrequency $\omega_i$ is not zero

$$\phi_{si}^T M_{ss}\phi_{si} + (1/\omega_i^2)(\phi_{ai}^T M_{as}\phi_{si} + \phi_{ai}^T M_{aa}\phi_{ai}) = 1 \tag{8}$$

When the eigenfrequency $\omega_i$ is zero

$$\phi_{ai}^T(M_{aa} + M_{as}K_{ss}^{-1}M_{as}^T)\phi_{ai} = 1 \qquad \text{for } \phi_{ai} \neq 0 \tag{9a}$$

$$\phi_{si}^T(M_{ss} + M_{as}^T K_{aa}^{-1}M_{as})\phi_{si} = 1 \qquad \text{for } \phi_{ai} = 0 \tag{9b}$$

## Mode-Superposition Technique for Coupled Acoustic-Structural Systems

By using the orthogonality condition of the coupled system given in the preceding section, the conventional mode-superposition method can be applied to the dynamic response analysis of the coupled system. First, the response of Eq. (1) is expressed with the lower eigenvectors of the coupled system as

$$u = \sum_{i=1}^{n} \phi_i q_i \tag{10}$$

where $n$ is smaller than the total number of modes of the system $N$. The solution thus obtained is substituted into Eq. (1) and the expression is premultiplied by $\bar{\phi}_i^T$. Then, using the orthogonality condition in Eq. (5), the following expression is obtained:

$$m_i \ddot{q}_i + k_i q_i = f_i, \qquad i = 1, 2, \ldots, n \tag{11}$$

When the eigenfrequency $\omega_i$ is not zero

$$m_i = \bar{\phi}_i^T M \phi_i = \phi_{si}^T M_{ss} \phi_{si} + \frac{1}{\omega_i^2}(\phi_{ai}^T M_{as} \phi_{si} + \phi_{ai}^T M_{aa} \phi_{ai}) \tag{12a}$$

$$k_i = \bar{\phi}_i^T K \phi_i = \phi_{si}^T K_{ss} \phi_{si} + \phi_{si}^T K_{sa} \phi_{ai} + \frac{1}{\omega_i^2} \phi_{ai}^T K_{aa} \phi_{ai} \tag{12b}$$

$$f_i = \bar{\phi}_i^T f = \phi_{is}^T f_s + \frac{1}{\omega_i^2} \phi_{ai}^T f_a \tag{12c}$$

When the eigenfrequency $\omega_i$ is zero

$$m_i = \begin{cases} \phi_{ai}^T (M_{aa} + M_{as} K_{ss}^{-1} M_{as}^T) \phi_{ai} & \text{for } \phi_{ai} \neq 0 \\ \phi_{si}^T (M_{ss} + M_{as}^T K_{aa}^{-1} M_{as}) \phi_{si} & \text{for } \phi_{ai} = 0 \end{cases} \tag{13a}$$

$$k_i = 0 \tag{13b}$$

$$f_i = \begin{cases} \phi_{ai}^T f_a & \text{for } \phi_{ai} \neq 0 \\ \phi_{si}^T (f_s - K_{sa} K_{aa}^{-1} f_a) & \text{for } \phi_{ai} = 0 \end{cases} \tag{13c}$$

In a frequency response analysis, the following expression is obtained from Eqs. (10) and (11) when $f = Fe^{i\Omega t}$ *and* $u = U e^{i\Omega t}$.

$$U = \sum_{i=1}^{n} \phi_i Q_i, \qquad Q_i = \frac{\phi_i^T F_i}{m_i(\omega_i^2 - \Omega^2)} \tag{14}$$

where $\Omega$ is the loading frequency

$$\omega_i = \sqrt{k_i/m_i} \text{ and } F_i = f_i e^{i\Omega t}$$

Equations (11) and (14) are identical in form to the mode-displacement equations used in structural dynamics. However, it should be noted that the calculation of the coefficients is different.

## Compensation Technique for Truncating Modes

In this section, a technique is derived for compensating the effect of the truncated modes that are not in the frequency domain of an MFR analysis. To provide better understanding of this problem, the mode-acceleration and the Hansteen-Bell's methods are first introduced.

### Mode-Acceleration Method

The mode-acceleration method can be used to accelerate the convergence of a solution in a dynamic response analysis. Also, by use of the revised orthogonality and normalization conditions of a coupled acoustic-structural system expressed previously, the mode-acceleration method can be applied to the coupled system. Here the mode-acceleration method is expressed based on the discussion of Ref. 9.

Assume that the mode-displacement approximate solution $\bar{u}$ is given by Eq. (10), where the modes from $(n + 1)$ to $N$ are completely ignored. The mode-acceleration solution is based on the following.

Equation (1) is written as

$$u = K^{-1}(f - M\ddot{u}) \tag{15}$$

and the $\ddot{u}$ term is approximated by the mode-displacement term $\ddot{\bar{u}}$. Thus, the mode-acceleration solution $u$ is given by

$$u = K^{-1}(f - M\ddot{\bar{u}}) \tag{16}$$

Substituting Eq. (10) into the right of Eq. (16) and incorporating Eq. (3) gives

$$u = K^{-1} f - \sum_{i=1}^{n} \frac{1}{\omega_i^2} \phi_i \ddot{q}_i \tag{17}$$

Furthermore, in an MFR analysis, the mode-acceleration solution $U$ is given by

$$U = K^{-1} F + \sum_{i=1}^{n} \left(\frac{\Omega}{\omega_i}\right)^2 \phi_i Q_i \tag{18}$$

The first term in Eq. (18) seems to be a static response, while the second term is in keeping with the name of the method. That is, if the error in Eq. (14) caused by neglecting the $i$th mode is $e_i^s = \phi_i Q_i$, then the error in Eq. (18) caused by neglecting same mode becomes

$$e_i^a = \left(\frac{\Omega}{\omega_i}\right)^2 e_i^s \tag{19}$$

It is seen in Eq. (19) that when $\omega_i > \Omega$, then $e_i^a < e_i^s$. Therefore, the error quadratically converges to zero.

It is also seen that if the truncated mode $i$ is lower than the loading frequency, the error $e_i^a$ becomes larger than the error of the mode-displacement method $e_i^s$ (Fig. 1). Therefore, the mode-acceleration method is not applicable to the truncation of the lower-frequency modes.

It should be noted that if stiffness matrix $K$ is singular, and therefore cannot be inverted, the mode-acceleration method cannot be employed in the straightforward manner indicated by Eq. (17) or Eq. (18). Reference 8 presents a complicated method to overcome this shortcoming.

### Hansteen-Bell's Method

Hansteen and Bell[10] pointed out that if the loading frequency is low, the effect of the higher-frequency modes can be approximated by a static analysis. For the sake of simplicity, we assume that the mass matrix $M$ is not singular‡ and $m_i = 1$. Then, Hansteen-Bell's method expresses the displacement solution $u$ of Eq. (1) as

$$u = \bar{u} + u_h \tag{20}$$

where $u_h$ is a residual unknown response, which expresses the contribution of the higher modes.

The excitation force term $f$ can also be expressed as a linear combination of eigenvectors, thus

$$f = \bar{f} + f_h \tag{21}$$

where

$$\bar{f} = \sum_{i=1}^{n} M \phi_i f_i \tag{22}$$

‡When $M$ is singular, one can reduce the DOF of the system by Guyan reduction[12] until $M$ is no longer singular.

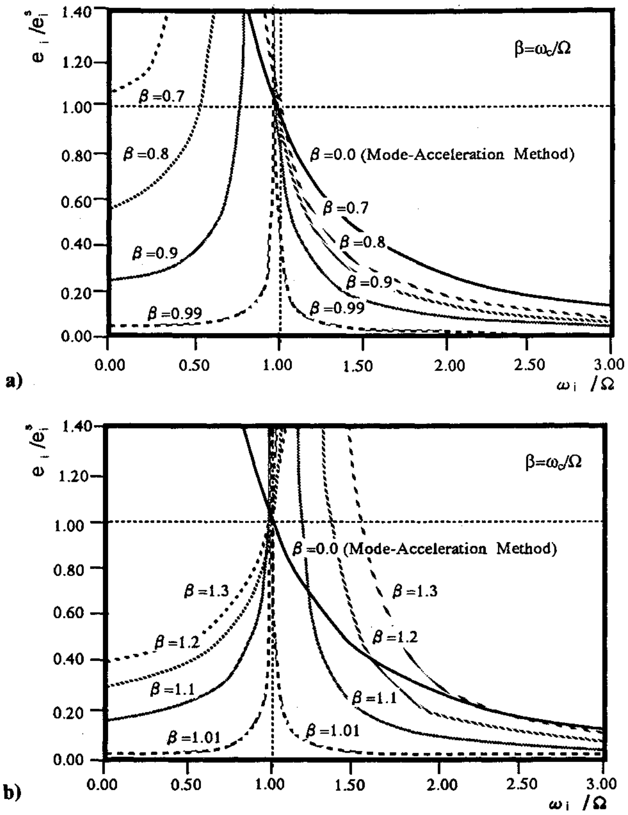


**Fig. 1 Comparison of computation errors caused by neglecting the *i*th mode; a) $\beta = 0.0$ and $\beta < 1$; and b) $\beta = 0.0$ and $\beta > 1$.**

is the component of $f$ corresponding to the lower modes, which is completely balanced by the approximate solution $\bar{u}$ (i.e., the equation $M\ddot{\bar{u}} + K\bar{u} = \bar{f}$ is exactly satisfied). And, $f_h = f - \bar{f}$ is a residual force, which is not balanced in the mode-displacement method. Using Eq. (22) and $f_i = \bar{\phi}_i^T f$, we can obtain

$$f_h = \left(I - \sum_{i=1}^{n} M\,\phi_i\bar{\phi}_i^T\right) f \tag{23}$$

In the Hansteen-Bell method, the residual force $f_h$ is balanced by the residual response $u_h$ with a static equilibrium equation:

$$K\,u_h = f_h \tag{24}$$

Substituting Eq. (23) into Eq. (24), and solving the resulting expression

$$u_h = K^{-1}\left(I - \sum_{i=1}^{n} M\,\phi_i\bar{\phi}_i^T\right) f \tag{25}$$

It can be proved that Hansteen-Bell's method is equivalent to the mode-acceleration method. In fact, using Eq. (3), Eq. (25) can be rewritten as

$$u_h = K^{-1} f - \sum_{i=1}^{n} \phi_i q_i^h \tag{26}$$

where

$$q_i^h = (1/\omega_i^2)\,\bar{\phi}_i^T f = (1/\omega_i^2)\,\ddot{q}_i + q_i \tag{27}$$

Accordingly, substituting $\bar{u} = \Sigma_{i=1}^{n} \phi_i q_i$ and Eq. (26) into Eq. (20), and incorporating Eq. (27), one obtains

$$u = K^{-1} f + \sum_{i=1}^{n} \phi_i(q_i - q_i^h) = K^{-1} f - \sum_{i=1}^{n} \phi_i \frac{1}{\omega_i^2}\ddot{q}_i \tag{28}$$

Since Eq. (28) is identical to Eq. (17), it is proved that Hansteen-Bell's method is equivalent to the mode-acceleration method.

### New Compensation Technique

As noted in the Introduction, an efficient acoustic-structural coupled analysis not only requires truncating the higher modes but also the lower ones. It is assumed that loading frequencies are considered within the range of $[\omega_a, \omega_b]$ $(\omega_a \le \omega_b)$, and that $m$ and $n$ are the numbers of the lowest and highest modes of coupled system used for an MFR analysis, respectively, where $m$ satisfies $\omega_m < \omega_a$ and $n$ satisfies $\omega_n > \omega_b$. Then, an accurate MFR solution can be written as

$$U = \sum_{i=m}^{n} \phi_i Q_i + U_r \tag{29}$$

where $U_r$ is an unknown residual MFR, which expresses the contribution of the truncated modes $\phi_i$ $(i = 1, \ldots, m-1, n+1, \ldots, N)$, i.e.,

$$U_r = \sum_{i=1}^{m-1} \phi_i Q_i + \sum_{i=n+1}^{N} \phi_i Q_i \tag{30}$$

and

$$Q_i = \frac{\bar{\phi}_i^T F}{m_i(\omega_i^2 - \Omega^2)}, \qquad i = 1, 2, \ldots, N \tag{31}$$

Assuming that $\phi_i$ is normalized corresponding to Eq. (8) or Eq. (9), then $m_i = 1$. Expanding Eq. (31) by the Taylor series on $\Omega^2 = \omega_c^2$ gives

$$Q_i = \frac{\bar{\phi}_i^T F}{\omega_i^2 - \omega_c^2}\left(1 + \frac{\Omega^2 - \omega_c^2}{\omega_i^2 - \omega_c^2} + \ldots\right)$$

$$\approx \frac{\bar{\phi}_i^T F}{\omega_i^2 - \omega_c^2} \qquad \text{for } |\Omega^2 - \omega_c^2| < |\omega_i^2 - \omega_c^2| \tag{32}$$

where $\omega_c \in [\omega_a, \omega_b]$ is a given constant frequency. Substituting Eq. (32) into Eq. (30),

$$U_r \approx GF = U_r' \tag{33}$$

where $G$ is called the residual flexibility matrix:

$$G = \sum_{i=1}^{m-1} \frac{\phi_i\bar{\phi}_i^T}{\omega_i^2 - \omega_c^2} + \sum_{i=n+1}^{N} \frac{\phi_i\bar{\phi}_i^T}{\omega_i^2 - \omega_c^2} \tag{34}$$

It is seen in Eq. (33) that the contribution of the truncated (lower and higher) modes, $U_r$, can be approximated by a quasistatic response $U_r'$, because $U_r'$ is independent of the loading frequency $\Omega$. However, the residual flexibility matrix $G$ in Eq. (33) cannot be calculated by Eq. (34), because usually the truncated modes are not calculated. Accordingly, we have to find $G$ using known quantities.

Letting the right and left eigenvector matrix be $\Phi = [\phi_1, \phi_2, \ldots, \phi_N]$ and $\bar{\Phi} = [\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_N]$, respectively, and the eigenvalue matrix be $\Lambda = \mathrm{diag}\{\omega_i^2\}$, the orthogonality and normalization conditions presented in the previous section can be rewritten as

$$\bar{\Phi}^T K \Phi = \Lambda, \qquad \bar{\Phi}^T M \Phi = I \tag{35}$$

Using Eq. (35) gives

$$(K - \omega_c^2 M) = \bar{\Phi}^{-T}\bar{\Phi}^T(K - \omega_c^2 M)\Phi\Phi^{-1}$$

$$= \bar{\Phi}^{-T}(\Lambda - \omega_c^2 I)\Phi^{-1} \tag{36}$$

and

$$(K - \omega_c^2 M)^{-1} = \bar{\Phi}(\Lambda - \omega_c^2 I)^{-1}\Phi^T = \sum_{i=1}^{N} \frac{\phi_i \bar{\phi}_i^T}{\omega_i^2 - \omega_c^2} \tag{37}$$

Therefore, the residual flexibility matrix can be obtained as follows:

$$G = (K - \omega_c^2 M)^{-1} - \sum_{i=m}^{n} \frac{\phi_i \bar{\phi}_i^T}{\omega_i^2 - \omega_c^2} \tag{38}$$

Substituting Eq. (38) into Eq. (33), and substituting the result into Eq. (29), gives

$$U = (K - \omega_c^2 M)^{-1} F + \sum_{i=m}^{n} \phi_i Q_i^d \tag{39}$$

where

$$Q_i^d = Q_i - \frac{\bar{\phi}_i^T F}{\omega_i^2 - \omega_c^2} = \frac{\Omega^2 - \omega_c^2}{\omega_i^2 - \omega_c^2} Q_i \tag{40}$$

It is shown that the MFR approximation is expressed as the sum of a quasistatic solution $U_s$, which corresponds to the following equation:

$$(K - \omega_c^2 M)U_s = F \tag{41}$$

and a residual dynamic response $U_d$, which is obtained by

$$U_d = \sum_{i=m}^{n} \phi_i Q_i^d = \sum_{i=m}^{n} \frac{\Omega^2 - \omega_c^2}{\omega_i^2 - \omega_c^2} \phi_i Q_i \tag{42}$$

i.e., $U = U_s + U_d$.

The error in Eq. (39) caused by neglecting $i$th mode is

$$e_i = \frac{\Omega^2 - \omega_c^2}{\omega_i^2 - \omega_c^2} e_i^s \tag{43}$$

The absolute error $|e_i|$ is smaller than that of the mode-displacement method $|e_i^s|$ provided that the convergence condition is satisfied, i.e.,

$$\omega_i^2 > \omega_c^2 + |\Omega^2 - \omega_c^2| \quad \text{or} \quad \omega_i^2 < \omega_c^2 - |\Omega^2 - \omega_c^2| \tag{44}$$

By setting $\omega_c = \sqrt{\frac{1}{2}(\omega_b^2 + \omega_a^2)}$, the foregoing conditions can be satisfied for all $\Omega \in [\omega_a, \omega_b]$, provided $i < m$ or $i > n$.

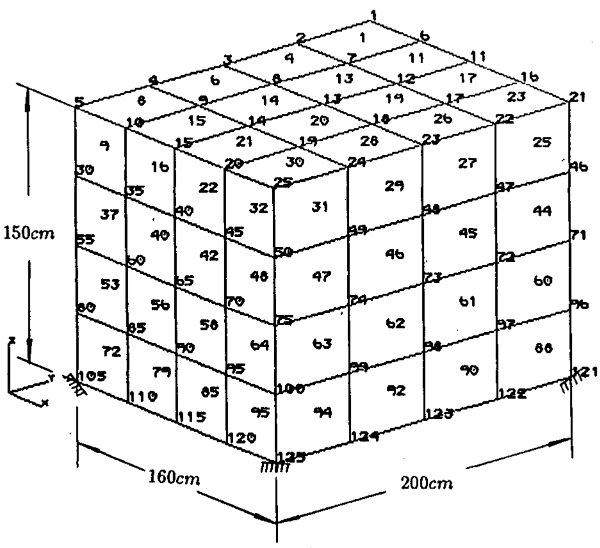


Fig. 2 Example of coupled acoustic-box structure.

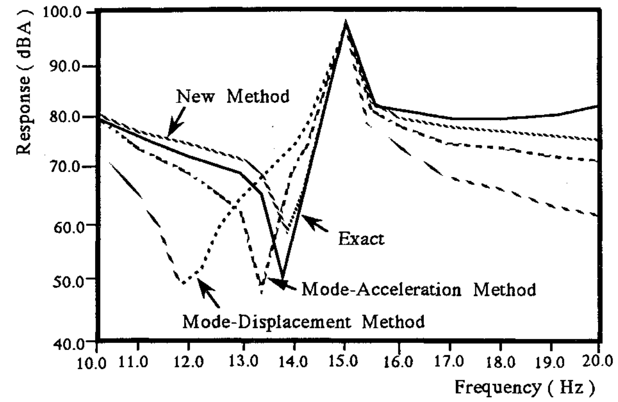


**Fig. 3 Comparison of MFR approximations with different methods when higher modes are omitted (using 1st–8th modes).**

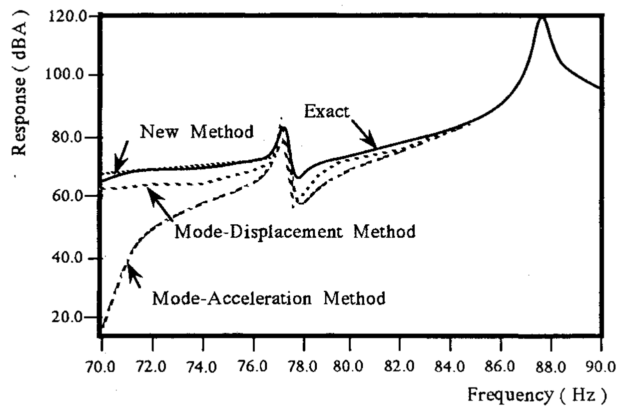


**Fig. 4 Comparison of MFR approximations with different methods when both lower and higher modes are omitted (using 30th–36th modes).**

Therefore, the convergence of the MFR approximation is accelerated, especially when $|\omega_i - \omega_c| >> 1$ and/or $\Omega \rightarrow \omega_c$.

It can be proved that for $m = 1$ and $n = N$, the MFR obtained from Eq. (39) is identical to the one obtained from Eq. (14), and, that for $m = 1$ and $\omega_c = 0$, the method proposed in this paper is reduced to the mode-acceleration method, which is equivalent to Hansteen-Bell's method.

Figure 1 shows a comparison of the convergence of the new method when the factor $\beta$, $\beta = \omega_c/\Omega$, is varied. Figure 1a shows the results when $\beta = 0.0$ and $\beta < 1$, and Fig. 1b shows the results when $\beta = 0.0$ and $\beta > 1$. The solid line is identical to the result obtained with the mode-acceleration method, i.e., $e_i = e_i^a$, when $\beta = 0.0$.

It is shown that the error with this method, $|e_i|$, is smaller than that of the mode-acceleration method, $|e_i^a|$, provided that $\omega_i/\Omega \geq 1$ or $\omega_i/\Omega \leq \beta/\sqrt{2 - \beta^2}$ (for $0 < \beta \leq 1$, Fig. 1a); $\omega_i/\Omega \leq 1$ or $\omega_i/\Omega > \beta/\sqrt{2 - \beta^2}$ (for $1 < \beta \leq \sqrt{2}$, Fig. 1b). For example, to truncate the higher modes, we can choose $\omega_c$ as $\omega_c = \omega_a$, which satisfies $\beta \leq 1$ for all $\Omega \in [\omega_a, \omega_b]$. Then the MFR solutions for all $\Omega \in [\omega_a, \omega_b]$ obtained by means of the new method are more accurate than those obtained with the mode-acceleration method.

Because the mode-acceleration method and also Hansteen-Bell's method do not apply to the truncation of the lower modes, the new method provides a substantial improvement over the former techniques, especially when the lower modes are truncated. Moreover, the mode-acceleration method and Hansteen-Bell's method are more complicated when the stiffness matrix $K$ is singular, whereas the new method requires no special treatment when $K$ is singular.

It should be noted that, from the convergence condition Eq. (44), the factor $\beta$ has to be selected as $\beta \geq 1/\sqrt{2}$, when lower modes are truncated. Finally, the factors determining

**Table 1 Comparison of SPL results for truncating higher modes** (dBA)

| Mode number | Mode displacement | Mode acceleration | This paper $\omega_c = 10$ Hz | This paper $\omega_c = 15$ Hz |
|---|---|---|---|---|
| | | a) $\Omega = 17$ Hz | | |
| 6 | 70.343 | 76.561 | 77.568 | 79.371 |
| 9 | 80.805 | 80.637 | 80.650 | 80.676 |
| 12 | 80.989 | 80.705 | 80.701 | 80.697 |
| 20 | 80.877 | 80.688 | 80.690 | 80.692 |
| 70 | 80.694 | 80.694 | 80.694 | 80.694 |
| | | b) $\Omega = 20$ Hz | | |
| 6 | 63.235 | 73.790 | 75.147 | 77.458 |
| 9 | 83.344 | 83.219 | 83.229 | 83.248 |
| 12 | 83.520 | 83.309 | 83.306 | 83.302 |
| 20 | 83.430 | 83.290 | 83.291 | 83.293 |
| 70 | 83.297 | 83.297 | 83.297 | 83.297 |

**Table 2 Comparison of SPL results for truncating lower and higher modes** (dBA)

| Mode number | Mode displacement | Mode acceleration | This paper $\omega_c = 80$ Hz | This paper $\omega_c = 85$ Hz |
|---|---|---|---|---|
| | | a) $\Omega = 82$ Hz | | |
| 7 (30–36) | 75.030 | 72.921 | 76.022 | 76.051 |
| 9 (28–36) | 75.035 | 72.931 | 76.021 | 76.052 |
| 17 (20–36) | 75.116 | 73.382 | 76.028 | 76.048 |
| 36 (1–36) | 75.481 | 75.781 | 76.011 | 76.072 |
| 70 (1–70) | 76.033 | 76.033 | 76.033 | 76.033 |
| | | b) $\Omega = 86$ Hz | | |
| 7 (30–36) | 88.617 | 88.216 | 88.834 | 88.840 |
| 9 (28–36) | 88.617 | 88.217 | 88.833 | 88.840 |
| 17 (20–36) | 88.635 | 88.297 | 88.836 | 88.840 |
| 36 (1–36) | 88.706 | 88.773 | 88.825 | 88.839 |
| 70 (1–70) | 88.842 | 88.842 | 88.842 | 88.842 |

the optimal value of the $\beta$ (i.e., the given frequency $\omega_c$) require further investigation.

## Application Examples

The validity of the MFR analysis method presented in this paper was verified using the coupled acoustic-box model shown in Fig. 2.[7] The coupled system was formed inside an empty rectangular parallelepiped made of steel plates and measuring 200 cm in length, 160 cm in width, and 120 cm in height. The box material had a Young's modulus of $2.1 \times 10^7$ Pa, density of $0.8 \times 10^{-6}$ kg/cm$^3$, and a Poisson ratio of 0.3. The thickness of the steel plates was 0.4 cm.

The structural model used in the analysis had 98 nodes and 96 quadrilateral plate elements (CQUAD4)[13] and the sound field had 125 nodes and 64 solid elements (CHEXA).[13] For the sake of simplicity, the physical coordinates of the structure and sound field were first converted to their respective modal coordinate systems before the analysis.[6] In effect, that meant expressing the overall coupled system in terms of 70 degrees of freedom, using 53 natural vibration modes of the structure, and 17 vibration modes of the sound field, including one rigid-body mode. To find the frequency response of the system, it was assumed that the excitation point was the 40th node along the $y$ axis of the box and that the measurement point was the 32nd node in the sound field.

Table 1 shows a comparison of the convergence of the new method presented in this paper, the mode displacement method, and the mode-acceleration method (Hansteen-Bell's method), when only higher modes of the coupled system were truncated. Table 1a shows the results of sound pressure level (SPL) when the loading frequency $\Omega$ was 17 Hz. As shown in the table, when six modes (between 0 Hz–20 Hz) were used, the errors of the mode-displacement method, mode-acceleration method, and the new method ($\omega_c = 10$ Hz and $\omega_c = 15$ Hz) were 12.8, 5.1, 3.9, and 1.6%, respectively. When 20 modes (between 0–49 Hz) were used, the error of the mode-displacement method was 0.2%, while the errors of the mode-acceleration method and new method became 0.007, 0.004, and 0.002%, respectively. It is shown that the new method has better convergence than the mode-displacement and the mode-acceleration methods. It is also shown that when the given frequency $\omega_c$ is close to the loading frequency $\Omega$, the convergence becomes more rapid. Table 1b shows the convergence when the loading frequency was 20 Hz. The results have the same trends.

Table 2 shows a comparison of the convergence of the new, the mode displacement, and the mode-acceleration methods when both lower and higher modes of the coupled system were truncated. Table 2a shows the results of SPL when the loading frequency $\Omega$ was 82 Hz. As shown in the table, when seven modes (between 77–110 Hz) were used, the errors of the mode-displacement method, mode-acceleration method, and the new method ($\omega_c = 80$ Hz and $\omega_c = 85$ Hz) were 1.3, 4.1, 0.01, and 0.02%, respectively. That is, the mode-acceleration method is not as accurate as the mode-displacement method, but the new method is more accurate than the mode-displacement method. Table 2b shows the results when the loading frequency was 86 Hz with similar results.

Figure 3 shows the frequency response data (SPL) obtained over a frequency range from 10–20 Hz with the mode-displacement method, mode-acceleration method, and the new method, respectively. The continuous line shows the exact solution. In this case, the first eight modes were used, and $\omega_c$ was given as 15 Hz. It is seen in the figure that the best results are obtained with the new method.

Figure 4 shows the SPL obtained over a frequency range from 70 to 90 Hz with the different methods. In this case, the 30th–36th modes were used, and $\omega_c$ was given as 80 Hz. It is seen in the figure that the mode-acceleration method is not as accurate as the mode-displacement method, and that the best result can be obtained with the new method.

## Conclusions

An improved mode-superposition technique with compensation for truncated modes was derived for obtaining the MFR of a coupled acoustic-structural system. It was shown that the new technique provides a substantial improvement over the mode-acceleration method and Hansteen-Bell's method. The new technique is effective in compensating for the effect of truncated higher modes as well as lower ones. This new technique greatly improves convergence and makes it possible to determine the MFR with good accuracy. Moreover, the new technique overcomes the inconvenience of the mode-acceleration and Hansteen-Bell methods when the stiffness matrix is singular.

A coupled acoustic-box model was used to confirm the superiority of the proposed technique to actual noise analysis problems.

The technique presented here could be developed into the instantaneous response analysis, response sensitivity analysis, model identification, component mode synthesis, etc. In the next step, the authors intend to investigate a method for conducting MFR sensitivity analyses of coupled acoustic-structural systems with the aim of reducing vehicle interior noise levels.

## Appendix

### Proof of the Orthogonality Conditions

For the sake of simplicity, it is assumed that Eq. (3) does not have repeated eigenfrequencies.

*1. When the Eigenfrequency $\omega_i$ Is Not Zero*

Premultiplying $(K - \omega_j^2 M)\phi_j = 0$ by $\bar{\phi}_i^T$ and using Eq. (6) results in

$$\bar{\phi}_i^T (K - \omega_j^2 M)\phi_j = \phi_{si}^T(K_{ss} - \omega_j^2 M_{ss})\phi_{sj} + \phi_{si}^T K_{sa}\phi_{aj} - (\omega_j^2/\omega_i^2)\,\phi_{ai}^T M_{as}\phi_{sj} + (1/\omega_i^2)\phi_{ai}^T(K_{aa} - \omega_j^2 M_{aa})\phi_{aj} = 0 \tag{45}$$

Postmultiplying $\bar{\phi}_i^T(K - \omega_i^2 M) = 0$ by $\phi_j$ and using Eq. (6) yields

$$\bar{\phi}_i^T (K - \omega_i^2 M)\phi_j = \phi_{si}^T(K_{ss} - \omega_i^2 M_{ss})\phi_{sj} + \phi_{si}^T K_{sa}\phi_{aj} - \phi_{ai}^T M_{as}\phi_{sj} + (1/\omega_i^2)\phi_{ai}^T \times (K_{aa} - \omega_i^2 M_{aa})\phi_{aj} = 0 \tag{46}$$

Subtracting Eq. (46) from Eq. (45) results in

$$(\omega_i^2 - \omega_j^2)[\phi_{si}^T M_{ss}\phi_{sj} + (1/\omega_i^2)(\phi_{ai}^T M_{as}\phi_{sj} + \phi_{ai}^T M_{aa}\phi_{aj})] = (\omega_i^2 - \omega_j^2)\bar{\phi}_i^T M\phi_j = 0 \tag{47}$$

Since $i \neq j$, $\omega_i^2 - \omega_j^2$ is not equal to zero; and the following expression must be true.

$$\bar{\phi}_i^T M\phi_j = 0 \qquad \text{for } i \neq j \tag{48}$$

Substituting Eq. (48) into Eq. (46) results in

$$\bar{\phi}_i^T K\phi_j = 0 \qquad \text{for } i \neq j \tag{49}$$

*2. When the Eigenfrequency $\omega_i$ Is Equal to Zero*

Since $\omega_i = 0$, the following expression can be obtained:

$$\bar{\phi}_i^T K = 0 \tag{50}$$

Premultiplying $(K - \omega_j^2 M)\phi_j = 0$ by $\bar{\phi}_i^T$ results in

$$\bar{\phi}_i^T K\phi_j = 0, \qquad \bar{\phi}_i^T M\phi_j = 0 \tag{51}$$

for $i \neq j$.

### Proof of the Normalization Conditions

By substituting Eqs. (6) and (7) into the M-normalization condition,

$$\bar{\phi}_i^T M\phi_i = 1 \tag{52}$$

respectively, and using $\phi_{si} = -K_{ss}^{-1}K_{sa}\phi_{ai}$, (for $\omega_i = 0$ and $\phi_{ai} \neq 0$), we obtain Eqs. (8) and (9).

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